

Metric Properties of Discrete Time Exclusion Type Processes in Continuum

Michael Blank

Received: 15 December 2008 / Accepted: 28 April 2010 / Published online: 11 May 2010
© Springer Science+Business Media, LLC 2010

Abstract A new class of exclusion type processes acting in continuum with synchronous updating is introduced and studied. Ergodic averages of particle velocities are obtained and their connections to other statistical quantities, in particular to the particle density (the so called Fundamental Diagram) is analyzed rigorously. The main technical tool is a “dynamical” coupling applied in a nonstandard fashion: we do not prove the existence of the successful coupling (which even might not hold) but instead use its presence/absence as an important diagnostic tool. Despite that this approach cannot be applied to lattice systems directly, it allows to obtain new results for the lattice systems embedding them to the systems in continuum. Applications to the traffic flows modelling are discussed as well.

Keywords Exclusion type process · Coupling · Ergodic average · Dynamical system

1 Introduction

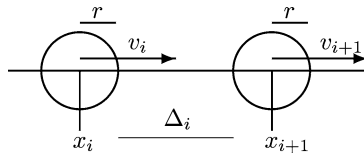
The classical simple exclusion process is a Markov process that describes nearest-neighbor random walks of a collection of particles on the one-dimensional infinite¹ integer lattice. Particles interact through the hard core exclusion rule, which means that at most one particle is allowed at each site. This seemingly very particular process introduced first in 1970 by Frank Spitzer [20] appears naturally in a very broad list of scientific fields starting from various models of traffic flows [5, 6, 10, 13, 16], molecular motors and protein synthesis in biology (see e.g. [21]), surface growth or percolation processes in physics (see [8, 18] for a review), and up to the analysis of Young diagrams in Representation Theory [9].

¹Or finite with periodic boundary conditions.

M. Blank (✉)
Inst. for Information Transm. Problems, Russian Academy of Sci., Moscow, Russia
e-mail: blank@iitp.ru

M. Blank
Laboratoire Cassiopee UMR6202, CNRS, Nice, France

Fig. 1 TASEP in continuum



Qualitatively from the point of view of the order of particle interactions there are two principally different types of exclusion processes: with synchronous and asynchronous updating rules. In the latter case at each moment of time a.s. at most one particle may move and hence only a single interaction may take place. This is the main model considered in the mathematical literature (see e.g. [14, 17, 20, 22] for a general account and [1, 11, 19] for recent results), and indeed, the assumption about the asynchronous updating is quite natural in the continuous time setting. The synchronous updating means that *all* particles are trying to move simultaneously and hence an arbitrary large (and even infinite) number of interactions may occur at the same time. This makes the analysis of the synchronous updating case much more difficult, but this is what happens in the discrete time case.² This case is much less studied, but still there are a few results describing ergodic properties of such processes [3, 5–7, 10, 13, 16].

Our aim is to introduce and study the synchronous updating version of the exclusion process in continuum. Note that recently some other interacting particle processes were generalized from lattice to continuum case (see e.g. [12, 18]).

A configuration $x := \{x_i\}_{i \in \mathbb{Z}}$ is a bi-infinite sequence of real numbers $x_i \in \mathbb{R}$ interpreted as centers of particles represented by balls of radius $r \geq 0$ (see Fig. 1) and ordered with respect to their positions (i.e. $\dots \leq x_{-1} \leq x_0 \leq x_1 \leq \dots$). To emphasize the dependence on the radius $r \geq 0$ we shall use the notation $x(r)$. We say that a configuration $x(r)$ is *admissible* if

$$x_i(r) + r \leq x_{i+1}(r) - r \quad \forall i \in \mathbb{Z}$$

(the corresponding balls may only touch each other) and denote by $X(r)$ the space of admissible configurations.

The dynamics will be defined as follows. We assume given a collection of (possibly random) values $\{v_i^t\}_{i,t}$, where $i, t \in \mathbb{Z}$ and $t \geq 0$; conditions on this collection will be given shortly. For a trivial configuration consisting of a single particle located at time $t \geq 0$ at $x_0^t \in \mathbb{R}$ (i.e. $x^t \equiv \{x_0^t\}$) the dynamics is defined as

$$x_0^{t+1} := x_0^t + v_0^t,$$

and thus v_0^t may be interpreted as a local velocity at time t , i.e. this is simply a random walk on \mathbb{R} . To generalize this trivial setting for an infinite configuration $x(r) \in X(r)$ we again interpret a (be-infinite on $i \in \mathbb{Z}$) sequence $\{v_i^t\}_{i,t}$ as *local velocities* for particles in $x^t(r)$ performing random walks conditioned to the order preservation and the hard core exclusion rule.

To simplify presentation we restrict ourselves here to the case of nonnegative local velocities postponing the discussion of the general case when the local velocities take both positive and negative signs to Sect. 6. The point is that the formulations in the latter case

²If one do not consider some “artificial” updating rules like a sequential or random updating.

are becoming much more involved, but the results and arguments work with only very slight changes.

Since only nonnegative local velocities are considered the hard core exclusion rule means that the admissibility condition breaks down for the i -th particle at time $t \in \mathbb{Z}_+$ if and only if the inequality

$$x_i^t(r) + v_i^t + r \leq x_{i+1}^t(r) - r$$

does not hold. If this happens we say that there is a *conflict* between the particles i and $i + 1$, and to resolve it one applies a *normalizing* construction

$$v_i^t \rightarrow \mathcal{N}(v_i^t, x^t(r)).$$

After the normalization the positions of particles are calculated according to the rule

$$x_i^{t+1}(r) := x_i^t(r) + \mathcal{N}(v_i^t, x^t(r)) \quad \forall i.$$

In what follows we always assume³ that $\forall i, t \ \mathcal{N}(v_i^t, x^t(r)) \in [0, v_i^t]$ (to simplify notation by the segment $[a, b]$ we mean $[\min(a, b), \max(a, b)]$) and consider only *nonanticipating* normalizations⁴ satisfying the condition that in the case of the conflict of the i -th particle with the j -th one⁵ at time t the position of the i -th particle at the next moment of time $x_i^{t+1}(r) \in [x_i^t(r), x_j^t(r)]$.

The normalization may be done in a number of ways and we restrict ourselves to two extreme constructions. The first of them we call *strong normalization* (notation $\mathcal{N}_s(\cdot, \cdot)$) and according to the name we reject (nullify) the velocity leading to the conflict. The second construction we call *weak normalization* (notation $\mathcal{N}_w(\cdot, \cdot)$) and in this case we modify the conflicting velocity to allow the particle to move as far as possible. In terms of *gaps*

$$\Delta_i(x^t(r)) \equiv \Delta_i^t := x_{i+1}^t(r) - x_i^t(r) - 2r$$

between particles in the configuration x^t the normalization procedures are written as follows:

$$\mathcal{N}_s(v_i^t, x^t(r)) := \begin{cases} v_i^t & \text{if } v_i^t \leq \Delta_i^t \\ 0 & \text{otherwise,} \end{cases} \quad \mathcal{N}_w(v_i^t, x^t(r)) := \begin{cases} v_i^t & \text{if } v_i^t \leq \Delta_i^t \\ \Delta_i^t & \text{otherwise.} \end{cases}$$

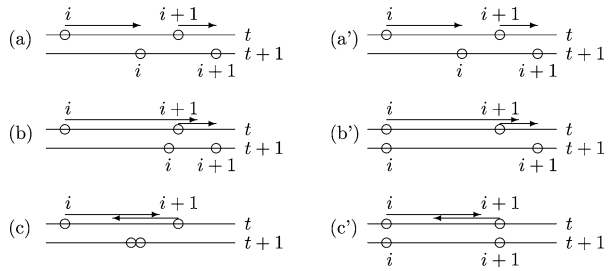
Figure 2 demonstrates possible positions of particles at two consecutive moments of time t and $t + 1$ for the cases of weak (a–c) and strong (a’–c’) normalizations. Despite appearances these two normalization procedures lead to a very different limit behavior of the corresponding particle systems. The simplest example (existing even in the continuous time case) is the situation when $v_i^t \equiv v \ \forall i, t$ and the gaps between particles in x are smaller than v . Then under the strong normalization no motion is allowed, while the weak normalization leads to the well defined motion—the exchange of gaps between particles. Other normalization procedures together with more general assumptions about the dynamics will be discussed in Sect. 7.

³This formulation allows to consider velocities of both signs which we shall do in Sect. 6 and simply means that the normalized velocity has the same direction as the original one and cannot exceed it on modulus.

⁴In Sect. 7 we shall show that the violation of this condition makes the system to be not well posed.

⁵For nonnegative velocities $j \equiv i + 1$, but in general $j \in \{i - 1, i + 1\}$.

Fig. 2 Positions of particles at time $t, t + 1$ in cases of weak (a–c) and strong (a'–c') normalizations. Local particle velocities are shown by vectors. The cases (c, c') correspond to negative velocities and will be discussed in Sect. 6



Observe that any two particle configurations $x(r), \hat{x}(\hat{r})$ having the same sequence of gaps $\Delta := \{\Delta_i\}$ may be transformed to each other by a one-to-one map

$$\hat{x}_i(\hat{r}) = \varphi(x_i(r)) := x_i(r) - 2i(r - \hat{r}) \quad \forall i \in \mathbb{Z}. \tag{1.1}$$

Since the normalization procedures that we consider depend only on the gaps between particles it is enough to study the case $r = 0$. On the other hand, if $r = 1/2, x_i^0(r) \in \mathbb{Z} \forall i \in \mathbb{Z}$ and $v_i^t \in \mathbb{Z} \forall i \in \mathbb{Z}, t \geq 0$ then $x_i^t(r) \in \mathbb{Z} \forall i \in \mathbb{Z}, t \geq 0$ which means that we get a lattice particle system. Thus our results lead to a completely new approach to the analysis of lattice systems as well. Note however that in the case $r = 0$ an arbitrary number of particles may share the same spatial position which is prohibited in the lattice case.

Due to the observation above we shall study in detail only the case $r = 0$ since the corresponding results for any $r > 0$ are readily available through the transformation (1.1), see e.g. specific calculations for densities and velocities in Lemmas 2.1, 2.4 and Corollaries 4.5, 5.3.

To simplify notation we shall use the convention $x(r) \equiv x^0(r), x \equiv x^0(0)$ and similarly $X \equiv X(0)$.

Of course, without some specific assumptions on the structure of local velocities $\{v_i^t\}_{i,t}$ no interesting results are possible. We assume that $v_i^t \in [0, v] \forall i \in \mathbb{Z}, t \in \mathbb{Z}_0 := \mathbb{Z}_+ \cup \{0\}$ and one of the following seemingly opposite assumptions holds:

- (a) $v_i^t \equiv v_0^t \forall i \in \mathbb{Z}, t \in \mathbb{Z}_0$ and $\exists \bar{v}(\gamma) := \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} \min(v_0^s, \gamma) \forall \gamma > 0$ (a.s.);
- (b) $\{v_i^t\}$ are i.i.d. (both in i and t) random variables.

Note that the intersection between the sets of local velocities satisfying the assumptions (a) and (b) contains an important case of pure deterministic velocities: $v_i^t \equiv v \forall i \in \mathbb{Z}, t \in \mathbb{Z}_0$. As we shall show properties of systems with local velocities satisfying to the assumption (a) are close to the pure deterministic setting. Therefore we refer to the setting (a) as *deterministic*⁶ and to the setting (b) as *random*.

It is of interest that in the seemingly simplest purely deterministic setting $v_i^t \equiv v \forall i \in \mathbb{Z}, t \in \mathbb{Z}_0$ the behavior of the corresponding deterministic dynamical system describing the dynamics of particle configurations is far from being trivial. In Sect. 4.3 we prove that this system is chaotic in the sense that its topological entropy is positive (and even infinite).

To emphasize that under dynamics no creation or annihilation of particles may take place this sort of systems is called *diffusive driven systems* (DDS) instead of a more general object—*interacting particle systems* (IPS).

⁶In this case v_0^t might be a trajectory of a deterministic chaotic map $f : [0, 1] \rightarrow [0, 1]$, e.g. $v_0^{t+1} := v f^t(v_0^t/v)$, as well as a realization of a true random Markov chain).

The main technical tool in our analysis is a (somewhat unusual) “dynamical” coupling construction. Despite that various couplings are widely used in the analysis of IPS, applications of our approach is very different from conventional. In particular, we do not prove the existence of the so called successful coupling (which even might not hold) but instead use its presence/absence as an important diagnostic tool. Remark also that typically one uses the coupling argument to prove the uniqueness of the invariant measure and to derive later other results from this fact. In our case there might be a very large number of ergodic invariant measures or no invariant measures at all (recall the trivial example of a single particle performing a skewed random walk). The latter example indicates that there is another important statistical quantity—average particles velocity that can be computed at least in this case. (See e.g. [2] for a discussion of the average velocity in the context of Queueing Networks.) The dynamical coupling will be used directly to find connections between the average particle velocities and other statistical features of the systems under consideration, in particular with the corresponding particle densities.

It is worth note that all approaches used to study lattice versions of DDS are heavily based on the combinatorial structure of particle configurations. This structure has no counterparts in the continuum setting under consideration. In particular the particle–vacancy symmetry is no longer applicable in our case. This explains the need to develop a fundamentally new techniques for the analysis of DDS in continuum. Despite this new techniques cannot be applied directly in the lattice case, the embedding of lattice systems to the continuum setting allows to obtain (indirectly) new results for the lattice systems as well.

The paper is organized as follows. In Sect. 2 we introduce main statistical quantities under study: particle densities, average velocities, etc. and derive their basic properties. Section 3 is dedicated to the main technical tool—dynamical coupling. In Sect. 4 we apply this coupling in the weak normalization setting to prove the uniqueness of the average velocity (Theorem 4.1) and to derive the complete Fundamental Diagram for the deterministic case (Theorem 4.2). We calculate also the topological entropy of this process (Theorem 4.3). The strong normalization case is considered in Sect. 5 (Theorem 5.1), while a more general setting with local velocities of both signs is studied in Sect. 6 (Theorem 6.1). Finally, in Sect. 7 we discuss some generalizations of our results and applications to certain specific traffic models.

2 Basic Properties of DDS

Here we shall study questions related to densities and velocities of DDS. To simplify notation we use the convention that the normalization $\mathcal{N} \in \{\mathcal{N}_s, \mathcal{N}_w\}$ and specify it only if this is necessary.

By the *density* $\rho(x, I)$ of a configuration $x \in X$ in a bounded segment $I = [a, b] \in \mathbb{R}$ we mean the number of particles from x whose centers x_i belong to I divided by the Lebesgue measure $|I| > 0$ of the segment I . If for any sequence of *nested* bounded segments $\{I_n\}$ with $|I_n| \xrightarrow{n \rightarrow \infty} \infty$ the limit

$$\rho(x) := \lim_{n \rightarrow \infty} \rho(x, I_n)$$

exists and does not depend on $\{I_n\}$ we call it the *density*⁷ of the configuration $x \in X$. Otherwise one considers upper and lower particle densities $\rho_{\pm}(x)$ corresponding to upper and lower limits.

⁷In Sect. 7.2 we shall show that this definition may be significantly weakened in the case when all particles move in the same direction.

The correspondence between particle densities for configurations with $r = 0$ and $r > 0$ is given by the following statement.

Lemma 2.1 *Let configurations $x(r) \in X(r)$, $r > 0$ and $x \in X$ have the same sequence of gaps $\{\Delta_i\}$. Then $\rho_{\pm}(x(r)) = \frac{\rho_{\pm}(x)}{1+2r\rho_{\pm}(x)}$.*

Proof Due to the one-to-one correspondence (1.1) between the configurations $x(r)$ and x , for each segment $I \subset \mathbb{R}^1$ which contains $\rho(x, I) \cdot |I|$ particles from the configuration x , one constructs the segment $I(r)$ containing the same particles from the configuration $x(r)$. The length of this segment is equal to $|I(r)| = |I| + 2r \cdot \rho(x, I) \cdot |I|$. Therefore

$$\rho(x(r), I(r)) = \frac{\rho(x, I) \cdot |I|}{|I| + 2r\rho(x, I) \cdot |I|} = \frac{\rho(x, I)}{1 + 2r\rho(x, I)}.$$

Passing to the limit as $|I| \rightarrow \infty$ one gets the result. □

Remark 2.2 If $\exists \rho(x) < \infty$ then $|x_n - x_m|/|n - m| \xrightarrow{|n-m| \rightarrow \infty} \rho^{-1}(x)$.

Lemma 2.3 *The upper/lower densities $\rho_{\pm}(x^t)$ are preserved by dynamics, i.e. $\rho_{\pm}(x^t) = \rho_{\pm}(x^{t+1}) \forall t \in \mathbb{Z}_0$.*

Proof For a given segment $I \in \mathbb{R}$ the number of particles from the configuration $x^t \in X$ which can leave it during the next time step cannot exceed 1 and the number of particles which can enter this segment also cannot exceed 1. Thus the total change of the number of particles in I cannot exceed 1, because if a particle leaves the segment through one of its ends no other particle can enter through this end. Therefore

$$|\rho(x^t, I) - \rho(x^{t+1}, I)| \cdot |I| \leq 1$$

which implies the claim. □

By the (average) *velocity* of the i -th particle in the configuration $x \in X$ at time $t > 0$ we mean

$$V(x, i, t) := \frac{1}{t} \sum_{s=0}^{t-1} \mathcal{N}(v_i^s, x^s) \equiv (x_i^t - x_i^0)/t.$$

If the limit

$$V(x, i) := \lim_{t \rightarrow \infty} V(x, i, t)$$

exists we call it the (average) *velocity* of the i -th particle. Otherwise one considers upper and lower particle velocities $V_{\pm}(x, i)$.

The correspondence between average particle velocities for configurations with $r = 0$ and $r > 0$ is even simpler than for densities.

Lemma 2.4 *Let configurations $x(r) \in X(r)$, $r > 0$ and $x \in X$ have the same sequence of gaps $\{\Delta_i\}$. Then $\forall i, t \ V(x(r), i, t) = V(x, i, t)$ for a given collection of local velocities $\{v_i^t\}_{i,t}$.*

Proof Observe that the motion of particles depends only on the local velocities and the sequence of gaps. Thus at any time $t \geq 0$ the sequence of gaps being changing in time is still the same for both configurations $x(r)$ and x . Therefore

$$\mathcal{N}(v_i^t, x^t(r)) \equiv \mathcal{N}(v_i^t, x^t) \quad \forall i, t$$

which yields the claim. □

Lemma 2.5 *Let $x \in X$ then $|V(x, j, t) - V(x, i, t)| \xrightarrow{t \rightarrow \infty} 0$ a.s. $\forall i, j \in \mathbb{Z}$.*

Proof It is enough to prove this result for $j = i + 1$. Consider the difference between (average) velocities of consecutive particles

$$\begin{aligned} V(x, i + 1, t) - V(x, i, t) &= \frac{x_{i+1}^t - x_{i+1}^0}{t} - \frac{x_i^t - x_i^0}{t} \\ &= \frac{x_{i+1}^t - x_i^t}{t} - \frac{x_{i+1}^0 - x_i^0}{t} \\ &= \Delta_i^t/t - \Delta_i^0/t. \end{aligned}$$

The last term vanishes as $t \rightarrow \infty$ and it is enough to show that the same happens with Δ_i^t/t .

Consider first the deterministic setting (i.e. $v_i^t \equiv v_0^t$) and show that⁸ $\forall i, t$

$$\Delta_i^t \leq \begin{cases} \max(v, \Delta_i^0) & \text{if } \mathcal{N} = \mathcal{N}_w \\ \max(2v, \Delta_i^0) & \text{if } \mathcal{N} = \mathcal{N}_s. \end{cases} \tag{2.1}$$

Obviously this is true for $t = 0$. Assume that this inequality holds up to time $t \in \mathbb{Z}_0$ and consider the moment $t + 1$. There might be two possibilities:

(a) $\Delta_i^t \geq v_0^t$. Then $\mathcal{N}(v_0^t, x^t) = v_0^t$ and

$$\Delta_i^{t+1} = \Delta_i^t - \mathcal{N}(v_i^t, x^t) + \mathcal{N}(v_{i+1}^t, x^t) \leq \Delta_i^t - v_0^t + v_0^t = \Delta_i^t \leq \max(v, \Delta_i^0)$$

by the assumption.

(b) $\Delta_i^t < v_0^t$. Then $\mathcal{N}_w(v_0^t, x^t) = \Delta_i^t$ and $\mathcal{N}_s(v_0^t, x^t) = 0$. Therefore

$$\begin{aligned} \Delta_i^{t+1} &= \Delta_i^t - \Delta_i^t + \mathcal{N}(v_{i+1}^t, x^t) \leq v \leq \max(v, \Delta_i^0) \quad \text{if } \mathcal{N} = \mathcal{N}_w, \\ \Delta_i^{t+1} &= \Delta_i^t - 0 + \mathcal{N}(v_{i+1}^t, x^t) \leq 2v \quad \text{if } \mathcal{N} = \mathcal{N}_s. \end{aligned}$$

Thus in the deterministic setting the gaps are uniformly bounded in time and hence $\Delta_i^t/t \xrightarrow{t \rightarrow \infty} 0$.

Analysis of the random setting is much more involved since the gaps between particles in principle may grow with time and become arbitrary large but this may happen only very slowly. To estimate from above the value of the i -th gap Δ_i^t we drop from the consideration all particles except the i -th and $(i + 1)$ -th (preserving for all $t \in \mathbb{Z}_0$ the velocities $\{v_i^t, v_{i+1}^t\}_t$)

⁸If v_0^t takes both positive and negative values then $\Delta_i^t \leq \max(4v, \Delta_i^0)$.

and denote the resulting configuration by $\tilde{x}^t := \{\tilde{x}_i^t, \tilde{x}_{i+1}^t\}$ and the gap between this pair of particles by $\tilde{\Delta}_i^t$. We have

$$\begin{aligned} \Delta_i^{t+1} &:= \Delta_i^t - \mathcal{N}(v_i^t, x^t) + \mathcal{N}(v_{i+1}^t, x^t), \\ \tilde{\Delta}_i^{t+1} &:= \tilde{\Delta}_i^t - \mathcal{N}(v_i^t, \tilde{x}^t) + \mathcal{N}(v_{i+1}^t, \tilde{x}^t) = \tilde{\Delta}_i^t - \mathcal{N}(v_i^t, \tilde{x}^t) + v_{i+1}^t. \end{aligned}$$

The comparison between Δ_i^t and $\tilde{\Delta}_i^t$ will be done by induction separately for the weak and strong normalizations.

First let us prove that $\tilde{\Delta}_i^t \geq \Delta_i^t$ if $\mathcal{N} = \mathcal{N}_w$. At time $t = 0$ obviously $\tilde{\Delta}_i^0 = \Delta_i^0$. Assume that $\tilde{\Delta}_i^t \geq \Delta_i^t$ for some $t \in \mathbb{Z}_+$. Clearly,

$$0 \leq \mathcal{N}(v_{i+1}^t, x^t) \leq v_{i+1}^t.$$

For v_i^t there might be two possibilities:

(a) $v_i^t \leq \Delta_i^t$. Then $\mathcal{N}(v_i^t, x^t) = \mathcal{N}(v_i^t, \tilde{x}^t) = v_i^t$ and hence

$$\tilde{\Delta}_i^{t+1} = \tilde{\Delta}_i^t - v_i^t + v_{i+1}^t \geq \Delta_i^t - v_i^t + v_{i+1}^t = \Delta_i^{t+1}.$$

(b) $v_i^t > \Delta_i^t$. Then $\mathcal{N}_w(v_i^t, x^t) = \Delta_i^t, \mathcal{N}_w(v_i^t, \tilde{x}^t) \geq \Delta_i^t$ and hence

$$\tilde{\Delta}_i^{t+1} = \tilde{\Delta}_i^t - \mathcal{N}_w(v_i^t, \tilde{x}^t) + v_{i+1}^t \geq v_{i+1}^t = \Delta_i^{t+1}.$$

If $\mathcal{N} = \mathcal{N}_s$ a weaker estimate $\tilde{\Delta}_i^t + v \geq \Delta_i^t$ takes place. Considering again the same possibilities we see that the cases $t = 0$ and (a) hold without any changes, but the case (b) should be rewritten.

(b') $v_i^t > \Delta_i^t$. Then $\mathcal{N}_s(v_i^t, x^t) = 0$,

$$\mathcal{N}_s(v_i^t, \tilde{x}^t) = \begin{cases} 0 & \text{if } v_i^t > \tilde{\Delta}_i^t \\ \tilde{\Delta}_i^t & \text{if } v_i^t \leq \tilde{\Delta}_i^t \end{cases},$$

and hence $\mathcal{N}_s(v_i^t, \tilde{x}^t) \geq \mathcal{N}_s(v_i^t, x^t)$. Thus

$$\begin{aligned} \tilde{\Delta}_i^{t+1} &= \tilde{\Delta}_i^t - \mathcal{N}_s(v_i^t, \tilde{x}^t) + v_{i+1}^t \\ &\geq \Delta_i^t - v - \mathcal{N}_s(v_i^t, x^t) + v_{i+1}^t - (\mathcal{N}_s(v_i^t, \tilde{x}^t) - \mathcal{N}_s(v_i^t, x^t)) \geq \Delta_i^{t+1} - v. \end{aligned}$$

Consider now the behavior of $\tilde{\Delta}_i^t$ as a function of time t . If $\tilde{\Delta}_i^t \geq v$ we get $v_i^t \leq \tilde{\Delta}_i^t$ and hence $\mathcal{N}(v_i^t, x^t) = v_i^t$, which implies that outside of the region $[0, v]$ the sequence $\tilde{\Delta}_i^t$ behave as a spatially homogeneous reflected at 0 random walk with i.i.d. symmetric increments $v_{i+1}^t - v_i^t$. Thus the mathematical expectation $E(\tilde{\Delta}_i^t)$ cannot exceed⁹ $2v$ and hence by Chebyshev inequality the probability

$$P(\tilde{\Delta}_i^t/t \geq \varepsilon) \leq \frac{1}{\varepsilon} E(\tilde{\Delta}_i^t/t) \leq \frac{2v}{t\varepsilon} \xrightarrow{t \rightarrow \infty} 0,$$

which finishes the proof. □

Corollary 2.6 *The upper and lower particle velocities $V_{\pm}(x, i)$ do not depend on i (but might be random).*

⁹ $4v$ if local velocities take both positive and negative values.

3 Coupling

Recall that a coupling of two Markov chains x^t and y^t acting on the space X is an arrangement of a pair of processes on a common probability space to facilitate their direct comparison, namely this is a pairs process (x^t, y^t) defined on the direct product space $X \times X$ satisfying the assumptions

$$P((x^t, y^t) \in A \times X) = P(x^t \in A) \quad \text{and} \quad P((x^t, y^t) \in X \times A) = P(y^t \in A)$$

for any measurable subset $A \subseteq X$, i.e. the projections behave as the individual processes.

Let x^t, \acute{x}^t be two copies of Markov chains, describing the DDS which we consider throughout the paper. Typically in continuous time interacting lattice particle systems one uses (see e.g. [14]) an *equal* coupling (pairing) when particles sharing the same sites in the copies x^t, \acute{x}^t are considered to be paired and all choices of their velocities are assumed to be identical. This sort of coupling works rather well for continuous time systems when only a single particle may move at a given moment of time. In the discrete time case the situation is much more complicated since an arbitrary number of particles may move simultaneously and thus it is possible that the particles of the processes x^t, \acute{x}^t pass each other and never share the same positions. In fact, this difficulty is not really crucial and can be cured under some simple technical assumptions. A more important obstacle is that if a pair is created and only one of its members is blocked by an unpaired particle, then due to the simultaneous motion of the blocking unpaired particle and the non-blocked particle belonging to the pair the following situation may happen: $\bullet^\circ \longrightarrow \circ^\circ$. Thus the old pair will be destroyed but no new pair will be created under the equal pairing construction. Here and in the sequel we use a diagrammatic representation for coupled configurations, where paired particles are denoted by black circles and unpaired ones by open circles, and use the upper line of the diagram for the x -particles (i.e. particles from the x -process) and the lower line for the \acute{x} -particles.

To deal with this obstacle we introduce a *dynamical*¹⁰ coupling, a very preliminary version of which was described in [7] for the lattice case and was inspired by the idea proposed by L. Gray for the simplest discrete time lattice TASEP (unpublished). It is worth mention also the coupling proposed for the lattice continuous time case by O. Angel (see [1, 11]). As we shall show an important advantage of the dynamical coupling with respect to the Angel's construction is that the former guarantees that the distances between mutually paired particles are uniformly bounded.¹¹

By the *dynamical coupling* of the processes x^t, \acute{x}^t we mean a gradual pairing of close enough particles belonging to the opposite processes satisfying the following assumptions:

- (A1) At $t = 0$ all particles are assumed to be unpaired. Velocities of mutually paired particles are identical.
- (A2) Once being created a pair of particles remains present¹² for any moment of time in the future, however at different moments of time the roles of the pair's members may be played by different particles.

¹⁰The word "dynamical" is meant to emphasize that the mutual arrangement of particles in pairs may change with time under dynamics in distinction to the conventional equal coupling (where the particles have coinciding positions).

¹¹In the Angel's construction the distances may grow to infinity.

¹²Starting from the moment when a pair is created we consider it as an entity independently on the possible change of particles forming it.

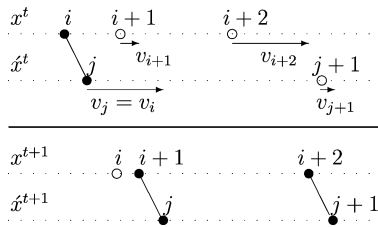


Fig. 3 Pairing of particles. Black circles corresponds to paired particles and open circles to defects. The paired particles are connected by straight lines. At time t the particles i and j are paired, while at time $t + 1$ the x -particle i becomes unpaired and the \hat{x} -particle j becomes paired with the x -particle $i + 1$. The unpaired initially particles $i + 2$ and $j + 1$ become paired at time $t + 1$

(A3) A particle overtaking during one time step of the dynamics some unpaired particles from the opposite process becomes paired with one of them.

According to (A1)–(A3) particles from the same pair move synchronously until either the admissibility condition breaks down for only one of the particles (which means that its movement is blocked by another particle) or one of the members of the pair is swapped with an unpaired particle from the same process (see Fig. 3 for the case of the weak normalization). It is convenient to think about the coupled process as a “gas” of single (unpaired) particles and “dumbbells” (pairs). A previously paired particle may inherit the role of the unpaired one from one of its neighbors. In order to keep track of positions of unpaired particles we shall refer to them as x - and \hat{x} -defects depending on the process they belong.

There are a number of ways to realize the dynamical coupling (in particular, using only the idea of the particle’s overtaking). To demonstrate the flexibility of our approach we describe a different construction. Note that in the sequel we shall use only the properties (A1)–(A3) and the proofs will not depend on other details of the coupling.

By the x -triple $(\bullet \circ \bullet \text{ or } \bullet \bullet \circ)$ in the coupled process (x^t, \hat{x}^t) we mean two mutually paired particles and a x -defect located in the segment between them, whose index differs by one from the index of the paired x -particle. The \hat{x} -triple $(\circ \bullet \bullet \text{ or } \bullet \bullet \circ)$ is defined similarly.

Two pairs of particles are said to cross each other if straight lines connecting positions of particles belonging to the same pair intersect, e.g. $\bullet \circ \bullet \circ$, where particles belonging to the same pair are marked similarly.

A x -defect at x_i^t together with the closest¹³ \hat{x} -defect at \hat{x}_j^t ($\circ \bullet$ or $\bullet \circ$) are said to be a d -pair if $|x_i^t - \hat{x}_j^t| < v$, this pair of defects does not cross with any mutually paired particles, and the open segment (x_i^t, \hat{x}_j^t) does not contain any other defects. We say that a d -pair (i, j) is smaller than a d -pair (n, m) if $|i| < |n|$, or if $i < n$ in case $|i| = |n|$. Observe that $i = n$ but $j \neq m$ cannot happen in distinction to $i \neq n$ but $j = m$.

Note that in the collection $\bullet \bullet \bullet$ the first two x -particles together with the first \hat{x} -particle form a x -triple despite the presence of an additional paired particle in the segment between them. On the other hand, the collection $\bullet \bullet \circ$ does not contain neither triples nor d -pairs.

A pair of configurations (x^t, \hat{x}^t) representing the coupled process at time t is said to be proper if it does not contain x - or \hat{x} -triples, d -pairs, and crossing mutually paired particles.

The fact that at time t the pair of configurations (x^t, \hat{x}^t) were proper does not imply that it remains proper under dynamics at time $(t + 1)$. In particular, triples of both types and d -pairs may be created, e.g. $\bullet \bullet \circ \rightarrow \bullet \bullet \bullet$ or $\circ \bullet \bullet \rightarrow \bullet \bullet \bullet$, however due to the particle order preservation crossing mutually paired particles cannot appear.

¹³If there are several closest \hat{x} -defects one chooses the defect with the smallest index.

Lemma 3.1 *Let a pair of configurations (x^t, \acute{x}^t) have no crossing mutually paired particles. Then among triples of the same kind there are no common elements.*

Proof Direct inspection. As an illustration let us check the claim about x -triples. Assume that two x -triples have a common x -defect (mutually paired particles cannot be common by definition). Then this implies that the mutually paired particles in these triples either cross each other $\circ \cdot \circ \cdot \circ$ or the index of one of the paired x -particles differs from the index of the common defect by more than one $\circ \cdot \circ \cdot \circ$. The latter contradicts to the definition of the x -triple, why the former contradicts to the assumption about the absence of crossing mutually paired particles. In the diagrams above paired particles from the 2nd triple are marked by stars to distinguish them from the 1st triple. □

Therefore all triples of the same kind may be resolved simultaneously. This will be done as follows. A x - or \acute{x} -triple is transformed such that the former defect is becoming paired to the particle from another process, while another previously paired particle is becoming unpaired: $\circ \cdot \circ \rightarrow \circ \cdot \circ$.

The case of a d-pair is even simpler, namely the defects “annihilate” forming mutually paired particles: $\circ \rightarrow \circ \cdot$. In all cases the positions of particles are preserved but their “roles” are changing.

Finally the coupling procedure consists of the following steps:

- (1) Each x -triple is recursively resolved: $\circ \cdot \circ \rightarrow \circ \cdot \circ$.
- (2) Each \acute{x} -triple is recursively resolved: $\circ \cdot \circ \rightarrow \circ \cdot \circ$.
- (3) The smallest¹⁴ d-pair is recursively resolved: $\circ \rightarrow \circ \cdot$.

Lemma 3.2 *The coupling procedure described above is well defined, leads to the Markovian coupling, and satisfies the assumptions (A1)–(A3).*

Proof Let us check that this procedure is well defined. By Lemma 3.1 if a particle belongs to a certain triple then it cannot belong to any other triple. On the other hand, segments belonging to paired particles may overlap and resolving a x - or \acute{x} -triple one may create a new one of the same kind:

$$\circ \cdot \circ \cdot \circ \rightarrow \circ \cdot \circ \cdot \circ \rightarrow \circ \cdot \circ \cdot \circ$$

This explains the necessity of the recursion during the first two steps of the procedure. Note that resolving a x -triple one cannot create a new \acute{x} -triple and vice versa (defects do not move from one process to another).

Elements of the smallest d-pair might belong to some other d-pairs. Therefore resolving it we might change the d-order of the remaining d-pairs. To take this into account we are recalculating the d-order after each recursion procedure.

Consider now the motion of a given defect under the recursions in the coupling procedure. Observe that the defect may move arbitrary far in any direction from its initial position due to these recursions:

$$\circ \cdot \dots \circ \cdot \circ \rightarrow \circ \cdot \dots \circ \cdot \circ$$

¹⁴The ordering of d-pairs is updated after each recursion procedure.

Nevertheless a defect cannot change its direction of movement. Assume from the contrary that a x -defect during two consequent steps of the recursion moved first to the right ($\circ \bullet \rightarrow \bullet \circ$) and then to the left ($\bullet \circ \rightarrow \circ \bullet$). This can happen only if after the first step of the recursion the defect became a member of a new x -triple of type $\bullet \circ \bullet$. Then the only candidate for the role of the paired x -particle in this x -triple is the paired x -particle which played the role of this defect on the previous recursion step. We came to the contradiction, because a particle may belong to only one pair.

Thus the recursion is finite in the sense that each defect in a bounded spatial segment in finite time either will stop moving or will leave this segment and never return back. Note however that in general one cannot divide a configuration into finite pieces and deal with them separately since a defect may move from one piece to another.

After the application of the first two steps all x - or \acute{x} -triples will be eliminated and only d-pairs may be present. Observe now that when one resolves a d-pair neither triples nor new defects are created. However since various d-pairs may intersect they should be resolved separately during the last step. Additionally neither of above procedures may create crossing pairs of mutually paired particles (since members of different triples of the same type do not intersect and c- and d-pairs cannot cross each other).

Let the pair of configurations $(x^{t-1}, \acute{x}^{t-1})$ be proper. Then according to arguments above after one time step of the dynamics the application of the coupling procedure, is well defined and the pair of configurations (x^t, \acute{x}^t) at time t is proper as well.

By the construction the one-time step transition probabilities for both processes x^t and \acute{x}^t remain unchanged and the one-time step transition probabilities for the pairs process are well defined. Therefore this construction defines a Markovian coupling between two copies of the Markov chain describing our DDS.

The property (A1) holds by the construction. A pair breaks down only if one of its members is replaced by an unpaired particle, and hence the pair as a whole survives. This proves (A2). The property (A3) follows from the fact that under the one time step of the dynamics of a proper pair of configurations all objects under consideration: x - and \acute{x} -triples, and d-pairs may be created only during the particles overtaking. □

Denote by $\rho_u(x, I)$ the density of the x -defects belonging to a finite segment I , and by $\rho_u(x) := \rho_u(x, \mathbb{R})$ the upper limit of $\rho_u(x, I_n)$ taken over *all* possible collections of nested finite segments I_n whose lengths go to infinity.

We say that a coupling of two Markov particle processes x^t, \acute{x}^t is *nearly successful* if the upper density of the x -defects $\rho_u(x)$ vanishes with time a.s. This definition differs significantly from the conventional definition of the successful coupling (see e.g. [14]), which basically means that the coupled processes converge to each other in finite time.

In the random setting under some regularity assumptions the dynamical coupling turns out to be nearly successful (the proof of this result goes out of the scope of the present paper and will be published elsewhere), however in general especially in the deterministic setting this property needs not hold.

Applying the notion of the nearly successful coupling to the exclusion process under study we get the following conditional result.

Lemma 3.3 *Let $x, \acute{x} \in X$ with $\rho(x) = \rho(\acute{x}) > 0$, and let there exist a nearly successful coupling (x^t, \acute{x}^t) such that distances between the pair members are uniformly bounded from above by $\gamma(t) = o(t)$. Then*

$$|V(x, 0, t) - V(\acute{x}, 0, t)| \xrightarrow{t \rightarrow \infty} 0.$$

Proof Consider an integer valued function n_t which is equal to the index of the \acute{x} -particle paired at time $t > 0$ with the 0-th x -particle. If the 0-th x -particle is not paired at time t we set

$$n_t := \begin{cases} n_{t-1} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

To estimate the growth rate of $|n_t|$ at large t observe that n_t changes its value only at those moments of time when the 0-th x -particle meets a \acute{x} -defect. By the assumption about the nearly successful coupling at time $t \gg 1$ the average distance between the defects at time t is of order $1/\rho_u(\acute{x}^t)$ while the amount of time needed for two particles separated by the distance L to meet cannot be smaller than $L/(2v)$. Therefore the frequency of interactions of the 0-th x -particle with \acute{x} -defects may be estimated from above by the quantity of order $\rho_u(\acute{x}^t) \xrightarrow{t \rightarrow \infty} 0$, which implies $n_t/t \xrightarrow{t \rightarrow \infty} 0$.

Now we are ready to prove the main claim.

$$\begin{aligned} |V(x, 0, t) - V(\acute{x}, 0, t)| &= |(x_0^t - x_0^0) - (\acute{x}_0^t - \acute{x}_0^0)|/t \\ &\leq |x_0^t - \acute{x}_0^t|/t + |x_0^0 - \acute{x}_0^0|/t \\ &\leq |x_0^t - \acute{x}_{n_t}^t|/t + \frac{|n_t|}{t} |\acute{x}_{n_t}^t - \acute{x}_0^t|/|n_t| + |x_0^0 - \acute{x}_0^0|/t. \end{aligned}$$

The 1st addend can be estimated from above by $\gamma(t)/t \xrightarrow{t \rightarrow \infty} 0$. The 2nd addend is a product of two terms $|n_t|/t$ and $|\acute{x}_{n_t}^t - \acute{x}_0^t|/|n_t|$. As we have shown, the 1st of them vanishes with time. If $|n_t|$ is uniformly bounded, then the 2nd term is obviously uniformly bounded on t . Otherwise, for large $|n_t|$ by Remark 2.2 and the density preservation the 2nd term is of order $1/\rho(\acute{x})$, which proves its uniform boundedness as well. Thus the 2nd addend goes to 0 as $t \rightarrow \infty$. Noting finally that the last addend also vanishes with time we are getting the result. □

4 Weak Normalization

Consider the coupled process (x^t, \acute{x}^t) under the weak normalization and set $W_{ij}^t := x_i^t - \acute{x}_j^t$.

Lemma 4.1 *The supremum of $|W_{ij}^t|$ taken over all mutually paired particles is uniformly bounded by v for any $t \in \mathbb{Z}_0$.*

Proof We start at time $t = 0$ when there are no pairs and wait until the first of them appears. At that moment the distance between the members in a pair cannot exceed v . Starting from that moment the distances may grow and some new pairs may be created. Contrary to our claim assume that there is the first moment of time t at which there is a pair of particles located at x_i^t, \acute{x}_j^t for which $|x_i^t - \acute{x}_j^t| > v$ and it is the largest distance between the paired particles at that moment of time (or one of the largest) and such that $|x_i^{t-1} - \acute{x}_j^{t-1}| \leq v$. According to the definition of the pairing process there are no unpaired particles between the particles from the same pair. Therefore in order to enlarge the distance between the particles one of them should be blocked by a particle from another pair, which contradicts to the assumption about the maximality of the distance. □

Lemma 4.2 *Let $\rho(x) = \rho(\acute{x})$ and let in the coupled process $\forall i, j \exists$ a (random) moment of time $t_{ij} < \infty$ such that $x_i^t > \acute{x}_j^t$ for each $t \geq t_{ij}$. Then the coupling is nearly successful.*

Proof By the assumption each x -particle will overtake eventually each \acute{x} -particle located originally to the right from its own position and thus will form a pair with it or with one of its neighbors (if they are so close that were overtaken simultaneously). Thus the creation of pairs is unavoidable. To show that the upper density of defects cannot remain positive, consider how the defects move under our assumptions. Assume that at time $t \geq 0$ the i -th x -particle is paired with the j -th \acute{x} -particle. Then by Lemma 4.1 in order to overtake at time $s > t$ the j -th \acute{x} -particle significantly (by a distance larger than v) the i -th x -particle necessarily needs to break the pairing with the j -th \acute{x} -particle. Thus by the property (A3) of the dynamical coupling either a x -defect overtakes the j -th \acute{x} -particle: $\circ \cdot \circ \rightarrow \circ \cdot \circ \rightarrow \cdot \circ \circ$, or the i -th x -particle overtakes a \acute{x} -defect: $\cdot \circ \circ \rightarrow \cdot \cdot \circ \rightarrow \circ \cdot \cdot$. (Otherwise this pair will not be broken.) Therefore during this process the x -defects move to the right while the \acute{x} -defects move to the left. Hence they inevitably meet each other and “annihilate”. The assumption about the equality of particle densities implies the result. \square

4.1 Uniqueness of the Average Velocity

As we shall see under our assumptions even in the weak normalization case the nearly successful coupling needs not hold (e.g. in the deterministic setting). Therefore one cannot apply directly Lemma 3.3 in this case. Nevertheless we shall show that the absence of coupling is not a serious obstacle and it can be used as a diagnostic tool.

Theorem 4.1 *In the weak normalization case the set of limit points as $t \rightarrow \infty$ of the sequence $\{V(x, t)\}_{t \in \mathbb{Z}_0}$ depends only on the density $\rho(x)$ assuming that the latter is well defined.*

Proof Consider a general DDS under the weak normalization. Let $x, \acute{x} \in X_\rho := \{z \in X : \rho(z) = \rho\}$ be two admissible configurations of the same particle density. If one assumes that the coupling procedure described in Sect. 3 leads to the nearly successful coupling of particles in these configurations then by Lemma 4.1 the assumptions of Lemma 3.3 are satisfied and hence $|V(x, 0, t) - V(\acute{x}, 0, t)| \xrightarrow{t \rightarrow \infty} 0$ which by Lemma 2.5 implies the claim. In general the assumption about the nearly successful coupling may not hold,¹⁵ however as we demonstrate below the pairing construction is still applicable.

Define random variables

$$W_{ij}^t := x_i^t - \acute{x}_j^t, \quad i, j \in \mathbb{Z}, t \in \mathbb{Z}_0.$$

Then

$$V(x, i, t) - V(\acute{x}, j, t) = W_{ij}^t/t - W_{ij}^0/t.$$

Since by Lemma 2.5 the differences between average velocities of different particles belonging to the same configuration vanish with time it is enough to consider only the case $i = j = 0$. For W_{00}^t there might be three possibilities which we study separately:

¹⁵Consider e.g. the deterministic setting with $1/\rho > 5v$ and the configurations $x_i := i/\rho$ and $\acute{x}_i := i/\rho + 2v$. Then $\rho(x) = \rho(\acute{x}) = \rho$, $V(x) = V(\acute{x}) = v$ but no pair will be created.

- (a) $\lim_{t \rightarrow \infty} W_{00}^t/t = 0$. Then $|V(x, 0, t) - V(\acute{x}, 0, t)| \leq |W_{00}^t|/t + |W_{00}^0|/t \xrightarrow{t \rightarrow \infty} 0$, which by Corollary 2.6 implies that the sets of limit points of the average velocities coincide.
- (b) $\limsup_{t \rightarrow \infty} W_{00}^t/t > 0$. Then $\forall i \in \mathbb{Z}$ the i -th particle of the x -process will overtake eventually each particle of the \acute{x} -process located at time $t = 0$ to the right from the point x_i^0 . This together with the assumption of the equality of particle densities allows to apply Lemma 4.2 according to which the coupling is nearly successful. On the other hand, by Lemma 4.1 the distance between mutually paired particles cannot exceed v . Therefore by Lemma 3.3 we have $|V(x, 0, t) - V(\acute{x}, 0, t)| \xrightarrow{t \rightarrow \infty} 0$, which contradicts to the assumption (b).
- (c) $\limsup_{t \rightarrow \infty} W_{00}^t/t < 0$. Changing the roles of the processes x^t, \acute{x}^t one reduces this case to the case (b).

Thus only the case (a) may take place. □

4.2 Deterministic Setting

Theorem 4.2 (Fundamental Diagram) *In the deterministic setting*

$$V(x) = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} \min(1/\rho, v_0^s) = \begin{cases} v & \text{if } \rho(x) \leq 1/v \\ 1/\rho(x) & \text{otherwise} \end{cases} \tag{4.1}$$

if $v_0^t \equiv v$.

Proof Consider a family

$$\check{X}_\rho := \{x \in X : x_i := i/\rho + \omega, \omega \in \mathbb{R}\}$$

of uniformly spatially distributed configurations of a given density $\rho > 0$. This set is forward invariant and

$$x_i^{t+1} - x_i^t \equiv \min(1/\rho, v_0^t) \quad \forall x^t \in \check{X}_\rho, i \in \mathbb{Z},$$

i.e. all particles in the configuration get the same normalized local velocity $\min(1/\rho, v_0^t)$ (depending in general on time t). By the definition of the deterministic setting the limit

$$V(x) := \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} \min(1/\rho, v_0^s)$$

is well defined. On the other hand, by Theorem 4.1 all configurations of the same density have the same average velocity, which implies the result. □

Remark 4.3 This result looks very similar to the one known for the deterministic version of the lattice TASEP (see [5, 16]), however the latter case is characterized by the following feature: if the density is large enough particles inevitably form dense clusters without vacancies inside (static traffic jams). The proof above shows that the “typical” behavior of high density configurations in continuum is different: they do form particle clusters, but these clusters are not staying at rest but are moving at a constant velocity as an “echelon”. It is of interest that in order to imitate such behavior a number of complicated lattice models were developed.

Remark 4.4 The construction used in the proof is especially striking in that the same family of uniformly spatially distributed configurations allows to study the limit dynamics in the deterministic setting for all configurations having densities. Note that this argument cannot be applied directly in the lattice version of DDS. Nevertheless since the “lattice configurations” are included in DDS under consideration the result holds as well, which implies completely new results for lattice TASEPs with long jumps.

Corollary 4.5 *Let $x(r) \in X(r)$, $r > 0$ and $\rho(x(r))$ be well defined and let $\forall i, t v_i^t \equiv v$. Then*

$$V(x(r)) = \begin{cases} v & \text{if } \rho(x) \leq \frac{1}{v+2r} \\ 1/\rho(x(r)) - 2r & \text{otherwise.} \end{cases}$$

In particular in the lattice setting this reads

$$V(x(1/2)) = \begin{cases} v & \text{if } \rho(x) \leq \frac{1}{v+1} \\ 1/\rho(x(1/2)) - 1 & \text{otherwise.} \end{cases}$$

Proof By (1.1) and Lemma 2.1 for each configuration $x(r)$ one constructs the configuration x with the same sequence of gaps and the relation between their densities is written as

$$\rho(x) = \frac{\rho(x(r))}{1 - 2r\rho(x(r))}.$$

Additionally by Lemma 2.4 average velocities related to configurations with the same sequence of gaps coincide. Substituting $\rho(x)$ as a function of $\rho(x(r))$ to (4.1) we get the result. □

4.3 Entropy

In this section we restrict the analysis to the pure deterministic setting (i.e. $v_i^t \equiv v \forall i, t$). Then our DDS is defined by a deterministic map $T_v : X \rightarrow X$ from the set of admissible configurations into itself. Our aim is to show that this map is chaotic in the sense that its topological entropy is infinite.¹⁶

We refer the reader to [4, 23] for detailed definitions of the topological and metric entropies for deterministic dynamical systems and their properties that we use here. To avoid difficulties related to the non-compactness of the phase space we define the topological entropy of a map T_v (notation $h_{\text{top}}(T_v)$) as the supremum of metric entropies of this map taken over all probabilistic invariant measures (compare to the conventional definition of the topological entropy and its properties in [23]).

For a finite subset of integers I and a collection $C := \{C_i\}_{i \in I}$ of open intervals the subset $C_{I,C} := \{x \in X : x_i \in C_i \forall i \in I\}$ is called a finite cylinder.¹⁷ We endow the space of admissible configurations X by the σ -algebra \mathcal{B} generated by the finite cylinders defining a topology in this space.

¹⁶Normally one says that a map is chaotic if its topological entropy is positive, so infinite value of the entropy indicates a very high level of chaoticity.

¹⁷In general the cylinder $C_{I,C}$ might be empty for nonempty sets I, C .

We start the analysis with the action of a shift-map in continuum $\sigma_v : X \rightarrow X$ defined as

$$(\sigma_v x)_i := x_i + v \quad i \in \mathbb{Z}, x \in X.$$

Lemma 4.6 *The topological entropy of the shift-map in continuum σ_v is infinite.*

Proof The preimage of a finite cylinder under the action of σ_v is again a finite cylinder. Therefore this map is continuous in the topology induced by the σ -algebra \mathcal{B} generated by finite cylinders.

The idea of the proof is to construct an invariant subset of X on which the map σ_v is isomorphic to the full shift-map in the space of sequences with a countable alphabet. The result follows from the observation that the topological entropy of the full shift-map $\sigma^{(n)}$ with the alphabet consisting of n elements is equal to $\ln n$ (see, e.g. [4, 23]).

Let $\alpha := \{\alpha_i\}_{i \in \mathbb{Z}_+}$ with $\alpha_i \in (0, v)$ and let $\alpha^n := \{\alpha_i\}_{i=1}^n$. Consider a sequence of subsets $X^{(n)} \subset X$ consisting of all configurations $x \in X$ satisfying the condition $\forall k \in \mathbb{Z} \quad x_{2k} \in v\mathbb{Z}, x_{2k+1} \in x_{2k} + \alpha^n$. Then $X^{(n)}$ is σ_v -invariant and the restriction $\sigma_v|X^{(n)}$ is isomorphic to the full shift-map $\sigma^{(n)}$ with the alphabet A^n consisting of n elements $\{a_i\}$ of type $a_i := \{[0, \alpha_i], [\alpha_i, v]\}$, i.e. each element is represented by a pair of neighboring intervals. Therefore the topological entropy of $\sigma^{(n)}$ is equal to $\ln n \xrightarrow{n \rightarrow \infty} \infty$. □

Another elegant (but technically difficult) way to derive this result was proposed by Boris Gurevich. Consider a special flow S^t corresponding to the shift-map acting on the sequences $\{\Delta_i(x)\}$ with the roof function equal to the first nonnegative particle coordinate. This shift-map has an infinite alphabet, hence its entropy is infinite. The special flow S^1 is isomorphic to the 1-shift of $\{x_i\}$, while the entropy of the special flow can be calculated by the Abramov-Rohlin formula.

Theorem 4.3 *The topological entropy of the pure deterministic exclusion process in continuum is infinite.*

Proof The preimage of a finite cylinder under the action of T_v is again a finite cylinder. Therefore this map is continuous in the topology induced by the σ -algebra \mathcal{B} generated by finite cylinders.

Observe that the subset $X_0 := \{x \in X : \Delta_i(x) \geq v \forall i \in \mathbb{Z}\}$ of the set of admissible configurations is T_v -invariant. Therefore $h_{\text{top}}(T_v) \geq h_{\text{top}}(T_v|X_0)$ and for our purposes it is enough to show that the latter is infinite. On the other hand, by the definition of the map T_v we have $T_v|X_0 \equiv \sigma_v|X_0$.

We still cannot apply the result of Lemma 4.6 directly because in the case under consideration the gaps between particles are greater or equal to v by the construction. Recall that in the proof of Lemma 4.6 the gaps were not greater than v . To this end one sets $\alpha_i \in (v, 2v)$ and modifies the definition of $X^{(n)}$ as follows:

$$x_{2k+1} \in x_{2k} + \alpha^n \quad \forall k \in \mathbb{Z}, x_{2k} \in 3v\mathbb{Z}.$$

Consider the alphabet $A^{(n)}$ with elements of type $a_i := \{[0, \alpha_i], [\alpha_i, 3v]\}$. Then the 3-d power of the map $T_v|X_0$ is isomorphic to the full shift-map $\sigma^{(n)}$ with the alphabet $A^{(n)}$. Using that

$$3h_{\text{top}}(T_v|X_0) = h_{\text{top}}((T_v|X_0)^3) = h_{\text{top}}(\sigma^{(n)}) = \ln n$$

we get the result. □

5 Strong Normalization

Recall that $W_{ij}^t := x_i^t - \hat{x}_j^t$ for $x^t, \hat{x}^t \in X, t \geq 0$.

Lemma 5.1 *There exists a coupled process (x^t, \hat{x}^t) such that under the strong normalization $\sup_{i,j,t} W_{ij}^t = \infty$, where the supremum is taken over all mutually paired particles.*

Proof It seems that the argument applied in the weak normalization case should work also in the case of the strong normalization. However, a close look shows that in this case a “blocked” particle does not move to “touch” the particle conflicting with it (as it would in the weak normalization case) but preserves its position instead. Therefore the distance between members of the same pair may become larger than the distance between the members of the “blocking” pair which cannot happen in the weak normalization case:
 $\begin{matrix} \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots \end{matrix} \longrightarrow \begin{matrix} \bullet & \bullet & \bullet \\ \vdots & \vdots & \vdots \end{matrix}$ Here initially distances between members in pairs do not exceed v . The 1st pair is blocked by the 2nd pair and since the \hat{x} -member of the 1st pair cannot move (while the x -member can) the distance between them becomes larger than v .

To demonstrate that distances between members in pairs may grow to infinity fix some $0 < \varepsilon \ll 1$ and consider a pair of configurations x, \hat{x} such that $x_0 = \hat{x}_0 = 0$ and $\Delta_{2k} = \frac{3}{2}(v - \varepsilon), \Delta_{2k+1} = \frac{1}{2}(v - \varepsilon), \hat{\Delta}_k = v - \varepsilon \forall k \in \mathbb{Z}$. After the application of the pairing procedure $\forall i$ the i -particles in both configurations will become paired forever. On the other hand, under dynamics $\hat{x}^t \equiv \hat{x}^0 \forall t$ while the x -particles having gaps greater than v will at constant velocity v . Therefore the distances between members in pairs will grow linearly with time. □

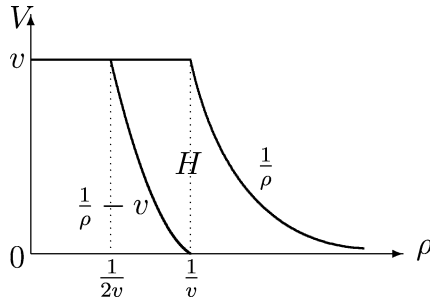
This result demonstrates and partially explains a significant difference in the behavior of DDS under weak and strong normalizations. Still, as we are going to show, at least some features of the Fundamental Diagram are preserved. Consider the pure deterministic setting (i.e. $v_i^t \equiv v$). The inequality (2.1) shows that in this case gaps between particles cannot become much larger than their initial values. The following result demonstrates that under some mild additional assumptions (which definitely hold for high particle densities) large gaps will disappear with time.

Lemma 5.2 *Let $x \in X$ be spatially periodic and we consider only the pure deterministic setting (i.e. $v_i^t \equiv v$). Assume that $\forall t \exists j > t : \Delta_j(x^t) < v$. Then $\forall i \exists t_i < \infty : \Delta_i(x^t) < 2v \forall t \geq t_i$.*

Proof Observe that the spatial periodicity and its period is preserved under the pure deterministic dynamics. Thus the situation is equivalent to the consideration of a finite number (say N) particles on a ring and to the assumption that for each $t \in \mathbb{Z}_+$ among these particles there is a particle with a gap less than v ahead of it. Note that according to the definition of the strong normalization $\mathcal{N}_s(v_i^t, x^t) = 0$ whenever $\Delta_i(x^t) < v$. By (2.1) $\Delta_i(x^t) < 2v$ implies $\Delta_i(x^{t+1}) < 2v$. Therefore new long gaps (of size larger or equal to $2v$) cannot be created and we need to show only that long gaps in the original configuration will cease to exist with time.

By the assumption for any t there exists a short gap (shorter than v) and the corresponding particle will not move during the next time step. Thus the index of the short gap decreases by one after each time step until it “collides” with one of the long gaps: $\Delta_i(x^t) \geq 2v, \Delta_{i+1}(x^t) < v$. On the next time step $\Delta_i(x^{t+1}) := \Delta_i(x^t) - v$. Due the spatial periodicity the amount of time between these “collisions” is bounded and after each of them the length of a long gap decreases by v . Thus they will disappear in finite time. □

Fig. 4 Fundamental Diagram (dependence of the average velocity V on the particle density ρ) for the pure deterministic setting under the strong normalization. The curvilinear region $H := \{(\rho, V) : \frac{1}{\rho} - v \leq V \leq \frac{1}{\rho}, V \leq v\}$ corresponds to the hysteresis phase



Theorem 5.1 Let $x \in X$ and $\rho(x)$ be well defined. Then $V(x) = v$ if $\rho(x) < \frac{1}{2v}$ and otherwise for a.e. point (ρ, V) in the curvilinear region

$$H := \{(\rho, V) : \max(1/\rho - v, 0) \leq V \leq \min(1/\rho, v)\}$$

(see Fig. 4) there exists a configuration $x \in X$ with $\rho(x) = \rho, V(x) = V$, i.e. the region H corresponds to the hysteresis.

Proof We say that particles numbered from $i + 1$ to $i + k$ with $i \in \mathbb{Z}, k \in \mathbb{Z}_+$ belonging to an admissible configuration $x \in X$ form a cluster of length k if all gaps between them are strictly less than v and the gaps to surrounding particles are not smaller than v , i.e. $\Delta_{i+j} < v \forall j = 1, 2, \dots, k - 1$ and $\Delta_i, \Delta_{i+k} \geq v$. Positions of particles belonging to the cluster are changing with time, and leading particles leave it, while some new particles may join the cluster from the other side. Nevertheless the length of a cluster cannot grow with time (and new clusters cannot be born in the pure deterministic setting in distinction to the random one) since the rate with which the leading particle leaves the cluster (one per unit time) is at least not smaller than the rate at which new particles join the cluster from the other side.

We start with the analysis of configurations of low density (smaller than $\frac{1}{2v}$) and our aim is to show that in this case each particle achieves eventually the largest available velocity v . Consider the motion of the 0-th particle in a configuration $x \in X$ with $0 < \rho(x) < \frac{1}{2v}$ and denote by \hat{t} the first moment of time after which this particle will not join any cluster. If $\hat{t} < \infty$ then $\mathcal{N}_s(v'_0) \equiv \forall t \geq \hat{t}$ and hence $V(x, 0, t) \xrightarrow{t \rightarrow \infty} v$.

If $\hat{t} = \infty$ then there exists an infinite sequence of clusters of growing length such that the 0-th particle joins each of them consecutively. Let us show that this assumption contradicts to the condition that $\rho(x) < \frac{1}{2v}$. We number the clusters to which the 0-th particle will join according to their natural order starting from $k = 1$ and introduce the following notation: t_k —the moment of time when the 0-th particle joins the k -th cluster, n_k —the number of particles in this cluster, m_k —the number of particles in the open segment between x_0 and the beginning of this cluster, and L_k —the length of the minimal segment containing the k -th cluster and the point x_0 . Then

$$\rho(x, (x_0, x_0 + L_k]) = \frac{m_k + n_k}{L_k} \xrightarrow{k \rightarrow \infty} \rho(x).$$

All m_k particles will join the k -th cluster during the time t_k and at time t_k this cluster should still exist. Therefore the distance which the 0-th particle covers during this time cannot be smaller than $L_k - m_k v - n_k v$ while its velocity cannot exceed v and thus

$$t_k v \geq L_k - m_k v - n_k v.$$

On the other hand, exactly t_k particles will leave the cluster during this time, i.e. $m_k + n_k \geq t_k$. This gives

$$\frac{m_k + n_k}{L_k} \geq \frac{t_k}{L_k} \geq \frac{L_k/v - m_k - n_k}{L_k} = \frac{1}{v} - \frac{m_k + n_k}{L_k}. \tag{5.1}$$

Therefore

$$\frac{1}{v} \leq 2 \frac{m_k + n_k}{L_k} \xrightarrow{k \rightarrow \infty} 2\rho(x),$$

which proves the desired claim that $\hat{t} = \infty$ implies $\rho(x) \geq \frac{1}{2v}$.

Consider now the case of densities greater than $\frac{1}{2v}$. In this case there might be two possibilities:

(a) All particles will eventually achieve the largest available velocity v . Then the gaps will become not smaller than v and hence they cannot exceed $2v$ (by the assumption on the density region). Obviously this situation may take place only if $\rho(x) \in [\frac{1}{2v}, \frac{1}{v}]$ and it corresponds to the upper branch of the Fundamental Diagram on Fig. 4.

(b) For any moment of time there are infinitely many particles having gaps smaller than v (and hence zero normalized local velocities). Therefore at least for spatially periodic configurations we can apply Lemma 5.2 which guarantees that only gaps smaller than $2v$ will survive with time. Thus to study asymptotic properties it is enough to consider configurations having only two types of gaps: smaller than v and between v and $2v$.

Denote by $X(L, m, n)$ the subset of admissible configurations $x \in X$ being spatially periodic with the spatial period of length $L \in \mathbb{R}_+$, which contains exactly $m \in \mathbb{Z}_+$ particles with gaps belonging to the interval $[0, v)$ and $n \in \mathbb{Z}_+$ particles with gaps belonging to the interval $[v, 2v)$. Obviously $\rho(x) = (m + n)/L$. The set $X(L, m, n)$ is invariant under dynamics (each time when the size of a gap crosses the threshold v one “small” gap becomes large and one “large” gap becomes “small”) which immediately yields the exact value of the average velocity $V(x) = \frac{nv}{m+n}$. On the other hand, by definition $mv + n2v > L$ since the corresponding gaps fill in the segment of length L and lengths of both types of gaps are smaller than v and $2v$ respectively. Therefore $(\rho(x)L + n)v > L$ and hence $n > L/v - \rho(x)L$, which gives the lower bound

$$V(x) = \frac{nv}{m+n} = \frac{nv}{\rho(x)L} > v \frac{L/v - \rho(x)L}{\rho(x)L} = 1/\rho(x) - v.$$

Observe, that choosing “small” and “large” gaps of length $v - \varepsilon$ and $2v - \varepsilon$ for $0 < \varepsilon \ll 1$ we see that the lower bound can be “almost” achieved.

The upper bound of the average velocity in the hysteresis phase (i.e. when $\frac{1}{2v} < \rho(x) < \frac{1}{v}$) follows from the existence of configurations with equal gaps of size larger than v for all densities from this segment. For the case $\rho(x) > 1/v$ the upper bound is calculated using the opposite length estimate $nv < L$. Then we get

$$V(x) = \frac{nv}{m+n} = \frac{nv}{\rho(x)L} < \frac{L}{\rho(x)L} = 1/\rho(x),$$

which agrees with the weak normalization case.

It remains to show that the region H is filled in densely by the pairs (ρ, V) corresponding to admissible configurations. To this end one considers all possible choices of the integer parameters n, m and lengths of the corresponding gaps to get the result. Indeed, $\forall \rho \in (\frac{1}{2v}, \frac{1}{v})$

there exists an arbitrary large L such that $\rho L \in \mathbb{Z}_+$. Choosing now various available combinations of positive integers m, n for which $m + n = \rho L$ we can approximate V with the accuracy

$$\left| V - \frac{nv}{m+n} \right| \leq \frac{v}{\rho L} \xrightarrow{L \rightarrow \infty} 0. \quad \square$$

Remark By Theorem 5.1 for a.e. pair $(V, \rho) \in H$ there exists an admissible configuration $x \in X$ such that $\rho(x) = \rho$ and $V(x) = V$. On the other hand, it might be possible that for some configurations having densities belonging to the hysteresis region the average velocity is not well defined and we claim only that all limit points of finite time velocities belong to the vertical segment corresponding to the given density.

Corollary 5.3 *Let $x(r) \in X(r)$, $r > 0$ and $\rho(x(r))$ be well defined and let $\forall i, t \ v_i^t \equiv v$. Then $V(x) = v$ if $\rho(x(r)) < \frac{1}{2v+2r}$ and otherwise for a.e. point (ρ, V) in the curvilinear region*

$$H := \left\{ (\rho, V) : \max\left(\frac{1}{\rho - 2r} - v, 0\right) \leq V \leq \min\left(\frac{1}{\rho - 2r}, v\right) \right\}$$

there exists a configuration $x(r) \in X(r)$ with $\rho(x(r)) = \rho, V(x(r)) = V$, i.e. the region H corresponds to the hysteresis.

6 Local Velocities of Both Signs

A close look to the previous analysis shows that we practically did not use the property that all particles move in the same direction, i.e. that $P(v_i^t \geq 0) = 1$. Now we explain the changes necessary to study this more general case. Consider an infinite configuration $x(r) \in X(r)$ and again interpret the values $\{v_i^t\}_{i,t}$ (which now may have both positive and negative signs, but still assuming that $|v_i^t| \leq v$) as local velocities for particles in the configuration $x^t(r)$.

The presence of particles moving in opposite directions leads to a serious modification of the inequalities describing the violation of the admissibility condition for the i -th local velocity. Actually this is the main and the most serious change comparing to the case of non-negative velocities. Now we need to take into account not only the position of the succeeding particle, but also its velocity, as well as the corresponding quantities related to the preceding particle. In this more general case the i -th local velocity does not break the admissibility condition if and only if

$$\begin{aligned} & \max(x_{i-1}^t(r), x_{i-1}^t(r) + v_{i-1}^t) + r \\ & \leq \min(x_i^t(r), x_i^t(r) + v_i^t) - r \\ & < \max(x_i^t(r), x_i^t(r) + v_i^t) + r \\ & \leq \min(x_{i+1}^t(r), x_{i+1}^t(r) + v_{i+1}^t) - r. \end{aligned}$$

If for some $i \in \mathbb{Z}$ and $j \in \{i - 1, i + 1\}$ the corresponding inequality is not satisfied we say that there is a *conflict* between the i -th particle and the j -th one and one needs to resolve it. In terms of gaps Δ_i^t between particles the inequalities above can be rewritten as follows:

$$\Delta_j^t \geq \max(v_j^t, -v_{j+1}^t, v_j^t - v_{j+1}^t), \quad j \in \{i - 1, i\} \tag{6.1}$$

Since the dynamics again will depend only on the sequence of gaps $\{\Delta_i^t\}$ between particles, for each $r > 0$ one can make the invertible change of variables (1.1) (described in the Introduction) to the case of ‘point’ particles with $r = 0$ which we shall study further.

Exactly as in Sect. 1 the *strong normalization* means that we reject (nullify) all velocities leading to a conflict, i.e.

$$\mathcal{N}_s(v_i^t, x^t) := \begin{cases} v_i^t & \text{if (6.1) holds} \\ 0 & \text{otherwise.} \end{cases}$$

The situation with the weak normalization is more delicate. The way how it was defined in Sect. 1 can be characterized as the only non-anticipating procedure allowing conflicting particles to move simultaneously whenever possible. Following this idea we say that a normalization is *weak* if the positions of particles at the next time step $x_i^{t+1} := x_i^t + \mathcal{N}_w(v_i^t, x^t)$ satisfy the conditions:

$$x_i^{t+1} \in \begin{cases} \{x_i^t + v_i^t\} & \text{if (6.1) holds} \\ \{x_j^t, x_j^{t+1}\} & \text{if } \exists \text{ a conflict of the particle } i \text{ with the particle } j = i \pm 1. \end{cases} \tag{6.2}$$

The 1st line describes the case when the admissibility condition holds, while the 2nd line shows what happens if it breaks down. Namely, if the i -th particle moves in the same direction as the j -th one then (by the non-anticipation property) the former assumes the previous position of the latter ($x_i^{t+1} = x_j^t$), otherwise the positions of the conflicting particles at time $t + 1$ coincide. The latter fact is the most important property here.

If directions of all instant local velocities coincide then (6.2) defines the normalization uniquely. However if their signs are different then (6.2) implies only that

$$x_i^{t+1} = x_j^{t+1} \in [x_i^t, x_j^t] \cap [x_i^t + v_i^t, x_j^t + v_j^t].$$

Thus the set of weak normalizations is quite broad, for example it includes a random normalization when two mutually conflicting particles moving in opposite directions meet at a random point belonging to the segments described above. One can give a ‘‘natural’’ specific construction of \mathcal{N}_w normalizing local velocities in such a way that positions of particles at the next moment of time will be the same as if the particles would move simultaneously at continuous time with the given local velocities until the admissibility condition breaks down:

$$\mathcal{N}_{w,c}(v_i^t, x^t) := \begin{cases} v_i^t & \text{if (6.1) holds} \\ -\Delta_{i-1}^t & \text{if } \Delta_{i-1}^t < -v_i^t, v_i^t < 0, v_{i-1}^t \leq 0 \\ \Delta_i^t & \text{if } \Delta_i^t < v_i^t, v_i^t > 0, v_{i+1}^t \geq 0 \\ \frac{\Delta_{i-1}^t}{v_{i-1}^t - v_i^t} \times v_i^t & \text{if } \Delta_{i-1}^t < v_{i-1}^t - v_i^t, v_i^t < 0, v_{i-1}^t > 0 \\ \frac{\Delta_i^t}{v_i^t - v_{i+1}^t} \times v_i^t & \text{if } \Delta_i^t < v_i^t - v_{i+1}^t, v_i^t > 0, v_{i+1}^t < 0. \end{cases}$$

After this long discussion of the definition of the normalization procedure it is surprising to find that all arguments used in the analysis of the case of positive velocities remain valid with only very slight changes.

Lemma 6.1 *The upper/lower densities $\rho_{\pm}(x^t)$ are preserved under dynamics.*

Proof One uses the same estimates as in the proof of Lemma 2.3 except that now 2 particles may simultaneously leave or enter a given spatial segment I (instead of 1). Thus the total change of the number of particles in I is less or equal to 2 and hence

$$|\rho(x^t, I) - \rho(x^{t+1}, I)| \cdot |I| \leq 2. \quad \square$$

Lemma 6.2 *Let $x \in X$ then $|V(x, j, t) - V(x, i, t)| \xrightarrow{t \rightarrow \infty} 0$ a.s. $\forall i, j \in \mathbb{Z}$.*

Proof Again one follows the same argument as in the case of nonnegative local velocities. The only difference is that in the analysis of the connection between Δ_i^t and $\tilde{\Delta}_i^t$ now one needs to consider new cases related to negative local velocities.

Additionally here instead of the uniquely defined weak normalization we need to consider an arbitrary one. If both v_i^t and v_{i+1}^t are nonnegative we are in the situation considered in Sect. 2. Therefore the cases (a) and (b) hold automatically. Nevertheless we formulate all of them to prove that $\tilde{\Delta}_i^t \geq \Delta_i^t \forall t \in \mathbb{Z}_0$:

- (a) the condition (6.1) holds. Then obviously the argument used in Sect. 2 woks.
- (b) $v_i^t > \Delta_i^t, v_{i+1}^t \geq 0$. Again one uses the same argument as in Sect. 2.
- (c) $v_i^t < -\Delta_{i-1}^t$. Then $\mathcal{N}_w(v_i^t, \tilde{x}^t) \leq \mathcal{N}_w(v_i^t, x^t) \leq 0$ and $\mathcal{N}_w(v_{i+1}^t, \tilde{x}^t) \geq \mathcal{N}_w(v_{i+1}^t, x^t)$.

Hence

$$\begin{aligned} \tilde{\Delta}_i^{t+1} &= \tilde{\Delta}_i^t - \mathcal{N}_w(v_i^t, \tilde{x}^t) + \mathcal{N}_w(v_{i+1}^t, \tilde{x}^t) \\ &\geq \Delta_i^t - \mathcal{N}_w(v_i^t, x^t) + \mathcal{N}_w(v_{i+1}^t, x^t) = \Delta_i^{t+1}. \end{aligned}$$

- (d) $v_i^t \geq 0, v_{i+1}^t < 0$ and $v_i^t - v_{i+1}^t > \Delta_i^t$. Then by definition $\tilde{\Delta}_i^{t+1} \geq 0 = \Delta_i^{t+1}$.

In the strong normalization setting one also considers the same cases and proves by induction that $\tilde{\Delta}_i^{t+1} \geq \Delta_i^{t+1} - 2v$ (instead of $\dots - v$ in the situation $v_i^t \geq 0$). New cases are the following

- (c') $v_i^t < -\Delta_{i-1}^t$. Then

$$\mathcal{N}_s(v_i^t, \tilde{x}^t) = v_i^t < -\Delta_{i-1}^t = \mathcal{N}_s(v_i^t, x^t)$$

and

$$\mathcal{N}_s(v_{i+1}^t, \tilde{x}^t) - \mathcal{N}_s(v_{i+1}^t, x^t) \geq -2v$$

by the induction assumption. Hence

$$\begin{aligned} \tilde{\Delta}_i^{t+1} &= \tilde{\Delta}_i^t - \mathcal{N}_s(v_i^t, \tilde{x}^t) + \mathcal{N}_s(v_{i+1}^t, \tilde{x}^t) \\ &> \Delta_i^t - 2v - \mathcal{N}_s(v_i^t, x^t) - \mathcal{N}_s(v_{i+1}^t, x^t) + 2v \\ &= \Delta_i^{t+1}. \end{aligned}$$

- (d') $v_i^t \geq 0, v_{i+1}^t < 0$ and $\Delta_i^t < v_i^t - v_{i+1}^t \leq \tilde{\Delta}_i^t$. Then

$$\begin{aligned} \tilde{\Delta}_i^{t+1} &= \tilde{\Delta}_i^t + v_i^t - v_{i+1}^t > \tilde{\Delta}_i^t + \Delta_i^t \\ &\geq -2v + \Delta_i^t = -2v + \Delta_i^{t+1}. \end{aligned}$$

(d'') $v_i^t \geq 0, v_{i+1}^t < 0$ and $\Delta_i^t < v_i^t - v_{i+1}^t > \tilde{\Delta}_i^t$. Then

$$\tilde{\Delta}_i^{t+1} = \tilde{\Delta}_i^t \geq \Delta_i^t - 2v = \Delta_i^{t+1} - 2v.$$

Note that the difference $\tilde{\Delta}_i^{t+1} - \Delta_i^{t+1} = -2v$ may be achieved only in the case (d'). The continuation of the proof is exactly the same as in Sect. 2, except for the change of $2v$ to $4v$ in the last inequality. □

Using these results and applying exactly the same arguments as in the proof of Theorem 4.1 one gets the uniqueness of the average velocity.

Theorem 6.1 *In the weak normalization case the set of limit points as $t \rightarrow \infty$ of the sequence $\{V(x, t)\}_{t \in \mathbb{Z}_0}$ depends only on the density $\rho(x)$.*

Theorem 6.2 (Fundamental Diagram) *In the deterministic setting $V(x) = \lim_{t \rightarrow \infty} \frac{1}{t} \times \sum_{s=0}^{t-1} \min(1/\rho, v_0^s)$.*

Proof Since at each moment of time $t \in \mathbb{Z}_0$ the local velocities of particles coincide, the condition (6.2) implies that

$$x_i^{t+1} \in \{x_i^t + v_0^t, x_{i \pm 1}^t\}.$$

Thus the construction used in the proof of Theorem 4.2 remains valid in this case as well. □

7 Generalizations and Discussion

7.1 Anticipating Normalization

Throughout the paper we consider only non-anticipating normalizations. In principle one might try to consider an anticipating normalization allowing at time t the i -th particle to move up to the position of the $(i + 1)$ -th particle x_{i+1}^{t+1} at time $t + 1$ rather than to x_{i+1}^t . From the first sight this makes the normalization scheme more flexible. Unfortunately the anticipating normalization is not well posed since it turns out to be nonlocal. Namely a single change in the sequence of local velocities (say of the i -th one) may drastically alter the behavior of the system for particles having indices arbitrary far from the changed one (i.e. for $j \ll i$).

7.2 One-sided Particle Densities

The density of a configuration in the way how it was defined in Sect. 2 depends sensitively on the statistics of both left and right tails of the configuration. A close look shows that in fact if all particles move in the same direction, say right, one needs only the information about the corresponding (right) tail, which allows to expand significantly the set of configurations having densities and for which our results can be applied.

For a configuration $x \in X$ by a *one-sided particle density* we mean the limit

$$\hat{\rho}(x) := \lim_{\ell \rightarrow \infty} \rho(x, [0, \ell]). \tag{7.1}$$

The upper and lower one-sided densities correspond to the upper and lower limits.

Theorem 7.1 *Let $v_i^t \geq 0 \forall i, t$. Then all results of Lemma 2.1 and Theorems 4.1, 4.2, 5.1 remain valid if one replaces the usual particle density ρ to the one-sided density $\hat{\rho}$.*

Proof The key observation here is that the assumption $v_i^t \geq 0 \forall i, t$ implies that the movement of a given particle in a configuration $x^t \in X$ depends only on particles with larger indices. Therefore if one changes positions of all particles with negative indices the particles with positive indices will still have the same average velocity. On the other hand, by Lemma 2.5 the average velocity does not depend on the particle index. This allows to apply the following trick.

For each configuration $x \in X$ of density $\rho(x)$ we associate a new configuration $\hat{x} \in X$ defined by the relation:

$$\hat{x}_i := \begin{cases} x_i^t & \text{if } i \geq 0 \\ x_0 + i/\rho(x) & \text{otherwise.} \end{cases}$$

Then obviously $\hat{\rho}(x) = \rho(\hat{x}) = \rho(x)$.

Therefore for all purposes related to the average velocities all results valid for the configuration \hat{x} remain valid for x as well. □

Note however that this trick does not work for the case of local velocities of both signs (considered in Sect. 6), nor in the passive tracer analysis (Sect. 7.4). In both these situations statistics of particles with negative indices cannot be neglected.

7.3 Nagel-Schreckenberg Traffic Flow Model

The celebrated Nagel-Schreckenberg traffic flow model introduced in [16] for the lattice case is very similar to our case but additionally to the lattice setting it uses a bit different dynamics. In our terms this model differs from the main model introduced in Sect. 1 by that at each time step the previous normalized local velocity of the i -th particle is increasing by $0 < a \leq a_i^t$ until it reaches v . One can think about a_i^t as an acceleration under the action of a (random) force (see e.g. [6]). Nevertheless the formalism elaborated in the present paper allows to study the continuum version of the Nagel-Schreckenberg model as well. In particular, in the weak normalization case one applies basically the same arguments as in Sects. 2, 3 and 4 since the distance between pair members cannot exceed $C(v, a) \leq v^2/a$. Note however that the average velocity should be calculated in a more complicated way. Observe also that one can consider random accelerations of both signs $a_i^t \in (-\infty, -a] \cup [a, \infty)$ which makes the model more applicable.

Mathematical formalism developed in the present paper can be applied with minimal changes to a number of other traffic flow models (discussed in detail, e.g. in a recent review [15]) allowing not only to study their continuum versions but also to get rigorous results in the original lattice setting which are absent at present.

7.4 Passive Tracer

Following the idea introduced in [5] we study the dynamics of a passive tracer in the flow of particles imitating a motion of a fast pedestrian in a slowly moving crowd of people.

Consider a pure deterministic setting ($v_i^t \equiv v$) with the weak normalization and let $T_v^t x$ describe the flow of particles. The passive tracer occupies the position $y^t \in \mathbb{R}$ at time t and moves all the time in the same direction. Before carrying out the next time step of the model

describing the flow of particles, the tracer moves in its chosen direction to the closest (in this direction) position of a particle of the configuration $T_v^t x$. After that the next iteration of the flow occurs, the tracer moves to its new position, etc.

To be precise, let us fix a configuration $x \in X$ with $\rho(x) > 0$ and introduce the maps $\tau_x^\pm : \mathbb{R} \rightarrow \mathbb{R}$ defined as follows:

$$\tau_x^+ y := \min\{x_i : x_i > y\}, \quad \tau_x^- y := \max\{x_i : x_i < y\}.$$

Then the simultaneous dynamics of the configuration of particles (describing the flow) and the tracer is defined by the skew product of two maps—the map T_v and one of the maps τ_x^\pm , i.e.

$$(x, y) \rightarrow \mathcal{T}_\pm(x, y) := (T_v x, \tau_x^\pm y),$$

acting on the extended phase space $X \times \mathbb{R}$. The sign $+$ or $-$ here corresponds to the motion along or against the flow. We define the *average (in time) velocity* of the tracer

$$V_{tr}(x, t) := (y^t - y^0)/t,$$

i.e. the total distance covered by the tracer (which starts at position $y^0 \in \mathbb{R}$) up to time $t \in \mathbb{Z}_+$ with the positive sign if the tracer moves forward, and the negative sign otherwise.

Theorem 7.2 *Let $v_i^t \equiv v \forall i, t$, $\mathcal{N} \equiv \mathcal{N}_w$, $x \in X$ and let $x_{i+1}^0 > x_i^0 \forall i \in \mathbb{Z}$. If the tracer moves along the flow (i.e. in the case \mathcal{T}_+) then*

$$V_{tr}(x, t) \xrightarrow{t \rightarrow \infty} V(x) = \begin{cases} v & \text{if } 0 < \rho(x) \leq 1/v \\ 1/\rho(x) & \text{otherwise.} \end{cases}$$

If the tracer moves against the flow (case \mathcal{T}_-) then $V_{tr}(x, t) \xrightarrow{t \rightarrow \infty} V(x) - 1/\rho(x)$.

Proof The assumption $x_{i+1}^0 > x_i^0 \forall i \in \mathbb{Z}$ implies that $x_{i+1}^t > x_i^t \forall i, t$ which allows to avoid a pathology related to the presence of several particles at the same position. In such a situation the tracer may “jump” through all of them in one time step. This cannot happen if $r > 0$ in distinction to the case of point particles ($r = 0$).

In the case of \mathcal{T}_+ the tracer will run down one of the particles in the flow and will follow it, but cannot outstrip. Thus $V_{tr}(x, t) \xrightarrow{t \rightarrow \infty} V(x)$.

Consider now the case when the tracer moves backward with respect to the flow, i.e. \mathcal{T}_- . Each time when the tracer encounters a particle, on the next time step this particle moves in the opposite direction and never will interfere with the movement of the tracer. Thus during time $t > 0$ the tracer meets exactly t particles which gives

$$(-V_{tr}(x, t) + V(x, t))t\rho(x) = t.$$

Therefore

$$V_{tr}(x, t) = -1/\rho(x) + V(x, t). \quad \square$$

Using similar arguments in the case of the strong normalization one can show that $V_{tr}(x, t)$ in the gaseous phase of the particle flow has the same asymptotic as in the weak normalization case. Since the flow in the fluid phase demonstrates hysteresis the same phenomenon is unavoidable for the passive tracer as well.

7.5 Multidimensional Generalization

The constructions used in this paper are essentially one-dimensional. Still at least some direct generalizations are possible. Let $x_i^t \in \mathbb{R}^d$, $d \in \mathbb{Z}_+$ and denote by $(x_i^t)_j$ the j -th coordinate of the d -dimensional vector x_i^t . We say that a configuration $x^t(r)$ is *admissible* if

$$\max_j((x_i^t(r))_j) + r \leq \min_j((x_{i+1}^t(r))_j) - r \quad \forall i \in \mathbb{Z}. \quad (7.2)$$

All results of Sects. 2, 3, 4.1, and 6 hold in this setting. Unfortunately the assumption (7.2) implies that a natural multidimensional generalization of the notion of density of the configuration $x^t(r)$ turns out to be equal to zero for any admissible configuration. However densities for one-dimensional projections are well defined and for them the Fundamental Diagram type results are readily available.

7.6 Open Problems and Conjectures

Our construction give a very precise information about the asymptotic properties of DDS under consideration in the deterministic setting. In the random setting we prove only the uniqueness of the average velocity. From the results of Sect. 2 it follows that the mathematical expectation of lower/upper average velocities are well defined but we are not able to calculate them. On the other hand, we can formulate a conjecture that the limits as time goes to infinity of finite time average velocities are deterministic. In other words, the Law of Large Numbers is valid for the sequence of finite time average velocities.

An important question is whether the dynamical coupling of pairs of processes with equal densities under the weak normalization is nearly successful. Let \mathcal{V} be the common distribution of the i.i.d. local velocities. As we know in the pure deterministic setting when the distribution \mathcal{V} is concentrated at a single point $\{v\}$ the dynamical coupling needs not to be successful. Nevertheless we conjecture that for each nontrivial distribution \mathcal{V} the nearly successful coupling takes place. Moreover, the non-triviality of the distribution \mathcal{V} should lead to the existence and uniqueness of the translationally invariant measure of the Markov chain described by the DDS. Proofs of results of this sort need the development of an additional probabilistic apparatus and will be discussed elsewhere.

Acknowledgements The author is grateful to B. Gurevich, S. Pirogov and an anonymous referee for a number of valuable remarks. This research has been partially supported by Russian Foundation for Fundamental Research, Program ONIT, and French Ministry of Education grants.

References

1. Angel, O.: The stationary measure of a 2-type totally asymmetric exclusion process. J. Comb. Theory Ser. A **113**(4), 625–635 (2006) [math.CO/0501005](https://arxiv.org/abs/math.CO/0501005)
2. Baccelli, F., Borovkov, A., Mairesse, J.: Asymptotic results on infinite tandem queueing networks. Probab. Theory Relat. Fields **118**(3), 365–405 (2000)
3. Belitsky, V., Ferrari, P.A.: Invariant measures and convergence for cellular automaton 184 and related processes. J. Stat. Phys. **118**(3–4), 589–623 (2005). [math.PR/9811103](https://arxiv.org/abs/math.PR/9811103)
4. Billingsley, P.: Ergodic Theory and Information. Wiley, New York (1965)
5. Blank, M.: Ergodic properties of a simple deterministic traffic flow model. J. Stat. Phys. **111**(3–4), 903–930 (2003). [math.DS/0206194](https://arxiv.org/abs/math.DS/0206194)
6. Blank, M.: Hysteresis phenomenon in deterministic traffic flows. J. Stat. Phys. **120**(3–4), 627–658 (2005). [math.DS/0408240](https://arxiv.org/abs/math.DS/0408240)

7. Blank, M., Pirogov, S.: On quasi successful couplings of Markov processes. *Probl. Pered. Inf.* **43**(4), 51–67 (2007) (Rus), pp. 316–330 (Eng). [math.DS/0603575](#)
8. Borodin, A., Ferrari, P.L., Sasamoto, T.: Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP. [arXiv:0707.4207](#) [math-ph]
9. Comtet, A., Majumdar, S.N., Ouvry, S., Sabhapandit, S.: Integer partitions and exclusion statistics: limit shapes and the largest parts of Young diagrams. *J. Stat. Mech.* P10001 (2007). [arXiv:0707.2312](#)
10. Evans, M.R., Rajewsky, N., Speer, E.R.: Exact solution of a cellular automaton for traffic. *J. Stat. Phys.* **95**, 45–98 (1999)
11. Evans, M.R., Ferrari, P.A., Mallick, K.: Matrix representation of the stationary measure for the multi-species TASEP. [arXiv:0807.0327](#) [math.PR]
12. Finkelshtein, D.L., Kondratiev, Y.G., Oliveira, M.J.: Markov evolutions and hierarchical equations in the continuum I. One-component systems. [arXiv:0707.0619](#) [math-ph]
13. Gray, L., Griffeath, D.: The ergodic theory of traffic jams. *J. Stat. Phys.* **105**(3–4), 413–452 (2001)
14. Liggett, T.M.: *Interacting Particle Systems*. Springer, New York (1985)
15. Maerivoet, S., De Moor, B.: Cellular automata models of road traffic. *Phys. Rep.* **419**(1), 1–64 (2005). [arXiv:physics/0509082](#)
16. Nagel, K., Schreckenberg, M.: A cellular automaton model for freeway traffic. *J. Phys. I* **2**, 2221–2229 (1992)
17. Nummelin, E.: *General Irreducible Markov Chains and Non-negative Operators*. Cambridge University Press, Cambridge (1984)
18. Penrose, M.D.: Existence and spatial limit theorems for lattice and continuum particle systems. *Probab. Surv.* **5**, 1–36 (2008)
19. Rosenthal, J.S.: Faithful couplings of Markov chains: now equals forever. *Adv. Appl. Math.* **18**, 372–381 (1997)
20. Spitzer, F.: Interaction of Markov processes. *Adv. Math.* **5**, 246–290 (1970)
21. Shaw, L.B., Zia, R.K.P., Lee, K.H.: Modeling, simulations, and analyses of protein synthesis: driven lattice gas with extended objects. *Phys. Rev. E* **68**, 021910 (2003)
22. Thorisson, H.: *Coupling, Stationarity, and Regeneration*. Springer, New York (2000)
23. Walters, P.: *An Introduction to Ergodic Theory*. Graduate Texts in Math., vol. 79. Springer, Berlin (1982). ix + 250 pp., ISBN 0-3879-0599-5